Mathematisches Institut
der Universität München
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WS 2014/15
– Blatt 11 –
17.12.2014

PDG I
(Zentralübung)

Problem Sheet 11

Question 1

Use the method of characteristics to determine the solution $u\colon [0,\infty)\times\mathbb{R}\to\mathbb{R}$ of the initial value problem

$$\begin{cases} u_t - xu_x = x & \text{in } (0,\infty)\times\mathbb{R} \\ u(0,x) = f(x) & \text{on } \{t=0\}\times\mathbb{R}\,, \end{cases}$$

where $f\in C^1(\mathbb{R})$ is given.

Question 2

Use the method of characteristics to find a solution $u\colon \mathbb{R}\setminus\{0\}\times\mathbb{R}\to\mathbb{R}$ $(=u(x,y))$ of the initial value problem

$$\begin{cases} (y+u)u_x + yu_y = x-y & \text{in } \mathbb{R}\setminus\{0\}\times\mathbb{R} \\ u(x,1) = 1+x & \text{on } \mathbb{R}\setminus\{0\}\times\{y=1\}\,. \end{cases}$$

Question 3

(a) Use the method of characteristics to find a solution $u=u(x,y)$ to the Cauchy problem

$$\begin{cases} x^2u_x + y^2u_y = u^2 \\ u(x,2x) = x^2 \end{cases}$$

(b) Check whether the transversatility condition holds.

Question 4

Suppose Ω is a bounded, open set in $\mathbb{R}^n$, $u\in C^1(\Omega)$ and $\varphi\in C_c^\infty(\Omega)$. Show that

$$\int_\Omega u\frac{\partial\varphi}{\partial x_i}\,\mathrm{d}x = -\int_\Omega \frac{\partial u}{\partial x_i}\varphi\,\mathrm{d}x\,.$$

(*Hint:* Note that we may not have $\partial\Omega$ is C^1 or $u\in C^1(\bar\Omega)$. But note that $u\varphi\in C_c^1(\Omega)$: so extend by zero to all of $\mathbb{R}^n$ and apply Gauss-Green to a ball containing Ω.)

Deadline for handing in: 0800 Wednesday 14 January

Please put solutions in Box 17, 1st floor (near the library)

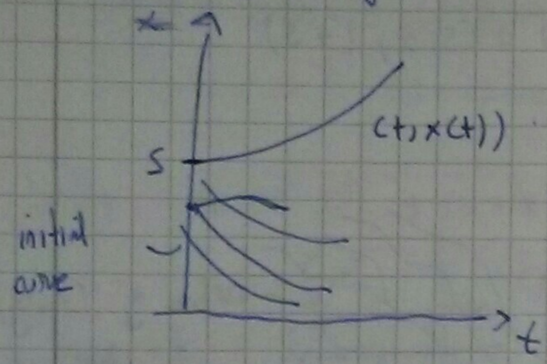
Homepage: `http://www.mathematik.uni-muenchen.de/~soneji/pde1.php`

Sheet 11

① Find a solution to the IVP

$$(1)\begin{cases} u_t - x u_x = x & \text{in } (0,\infty)\times\mathbb{R} \\ u(0,x) = f(x) & \text{on } \{t=0\}\times\mathbb{R} \end{cases}$$

where $f \in C^1(\mathbb{R})$ given.



Fix $s \in \mathbb{R}$ and consider the curve $(t, x(t))$ in $[0,\infty)\times\mathbb{R}$,

where $x : [0,\infty) \to \mathbb{R}$ is C^1. Now define

$z(t) = u(t, x(t))$, where u is a solution to (1).

Then $z'(t) = u_t(t, x(t)) + x'(t)\, u_x(t, x(t))$

If $(*)\ x'(t) = -x(t)$, ~~then~~ – characteristic equation

then $z'(t) = x(t)$ since u solves (1)

Solve (2): $x(t) = C_1 e^{-t}$

$x(0) = s$, so $x(t) = se^{-t}$

So $z'(t) = x(t) = se^{-t}$

$z(t) = se^{-t} + C_2$.

$z(0) = u(0, x(0)) = u(0,s) = f(s). = s + C_2.\quad C_2 = f(s) - s$

So $z(t) = se^{-t} + f(s) - s. = u(t, x(t))$

$= u(t, se^{-t})$ – parametrised solution of u

Let $(t,x) \in [0,\infty)\times\mathbb{R}$.

Then want s s.t. $x = se^{-t}$. ie $s = xe^{t}$.

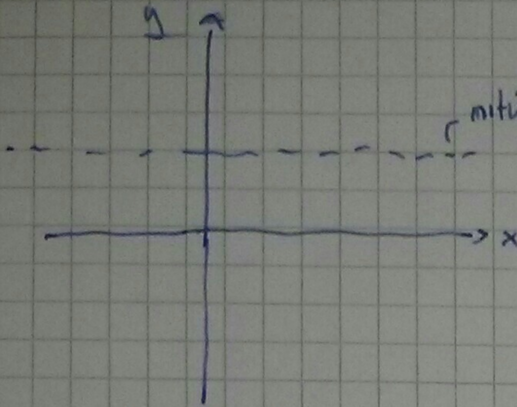
Then and so solution to u gives $u(t,x)$

$$u(t,x) = se^{-t} + f(s) - s = (xe^{t})e^{-t} + f(xe^{t}) - xe^{t}$$

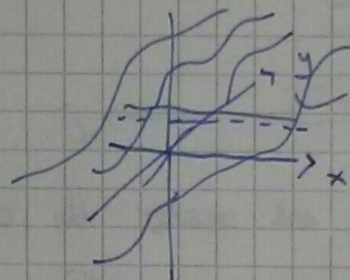
$$= x(1-e^{t}) + f(xe^{t})$$

② Find a solution to

$$(3)\begin{cases} (y+u)u_x + y u_y = x-y & \mathbb{R}\times\mathbb{R} \\ u(x,1) = 1+x & \mathbb{R}\times\{y=1\} \end{cases}$$

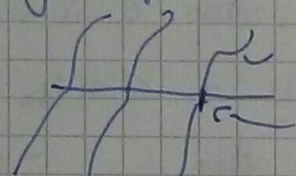


initial curve $\Gamma(s) = (s, 1, 1+s)$



surface of $(x, y, u(x,y)) = S$
where u solves (2)

Consider curve along surface S ~~$(x(t), y(t), u(x(t), y(t)))$~~ $u(t)$



$\Gamma(s)$

$(x(t,s), y(t,s), u(t,s))$

where at $t=0$, this curve intersects Γ at $\Gamma(s)$. Direction of ~~At point A tangent to~~ this curve given by

(x_t, y_t, u_t). But $(u_x, u_y, -1)$ is normal to S.

^ in tangent plane of S.

Here if we solve $(x_t, y_t, u_t) = (y+u, y, x-y)$ – characteristic equations

Then we can get a representation for u in terms of s, t (and then (x,y)).

$$\begin{cases} x_t(t,s) = y(t,s) + u(t,s) \\ y_t(t,s) = y(t,s) \quad . \text{ So } \quad y(t,s) = c_1 e^t. \quad y(0,s) = 1. \text{ So } y(t,s) = e^t \\ u_t(t,s) = x(t,s) - y(t,s) \end{cases}$$

So $\quad x_t = e^t + u$

$$u_t = x - e^t$$

$$(u+x)_t = u + x$$

So $\quad u + x = c_2 e^t$

$$u(0,s) + x(0,s) = (1+s) + s = 1+2s$$

So $\quad u + x = (1+2s)e^t$

Also, $\quad x_t = e^t + (u+x) - x$

$$= e^t + (1+2s) - x$$

$$x_t + x = 2(1+s)e^t$$

Integrating factor, e^t: standard ODE: $\quad x(t,s) = (1+s)e^{\cancel{x}t} + c_3 e^{-t}$

$$x(0,s) = s = (1+s) + c_3$$

So $\quad x(t,s) = (1+s)e^t - e^{-t}$

Hence $\quad u_t = \cancel{(1+s)e^t - e^{-t}} - e^t$

$$\cancel{= se^t - e^{-t}}$$

~~So $u(t,s) =$~~

$$u(t,s) = (u+x) - x = (1+2s)e^t - (1+s)e^t + e^{-t}$$

$$= se^t + e^{-t}$$

Recall $y(t,s) = e^t$.

So $\quad x - y = se^t - e^{-t}$

Hence ~~$u(t,s)$~~ $\quad u = (x-y) + \frac{2}{y}$

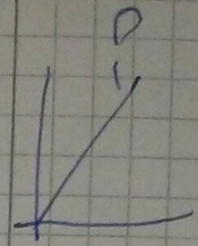
Not a global solution singular on $\{y=0\}$

but well defined in a neighbourhood of $\mathbb{R}\times\{y=1\}$.

③ (a) Find a solution to Cauchy Problem

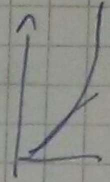
$$(4)\quad \begin{cases} x^2 u_x + y^2 u_y = u^2 \\ u(x, 2x) = x^2 \end{cases} \qquad u = u(x, y)$$

Initial Curve $\Gamma(s) = (s, 2s, s^2)$



Characteristic equations are:

$$\begin{cases} x_t = x^2 & x(0,s) = s \\ y_t = y^2 & y(0,s) = 2s \\ u_t = u^2 & u(0,s) = s^2 \end{cases}$$



General soln to $v_t = v^2$ is $v(t) = \dfrac{1}{C - t}$

Using initial conditions at $t = 0$, we get:

$$x(t,s) = \frac{s}{1-ts} \qquad y(t,s) = \frac{2s}{1-2ts} \qquad u(t,s) = \frac{s^2}{1-ts^2}$$

Solve to express u in terms of x, y instead of t, s.

$(1-ts)x = s$, so $x - tsx = s$ so $tsx = x - s$

Hence $t = \dfrac{x-s}{xs}$ (provided $x, s \neq 0$)

$\cancel{(1-2st)y = 2s}$

so $s = \dfrac{y}{2(1+ty)} = \dfrac{y}{2\left(1+\left(\frac{x-s}{xs}\right)y\right)}$

$= \dfrac{xs}{2xs + 2(x-s)y}$

So Also $(1-2ts)y = 2s$

$\left(1 - 2\left(\frac{x-s}{xs}\right)s\right)y = 2s$

$(x - 2(x-s))y = 2sx$

$xy - 2xys + 2ys = 2sx$

$2s(x + xy - y) = xy$ $\qquad s = \dfrac{xy}{2(1+y(x-1))}$

Also, $(1-2ts)y = 2s$

so $\left(1-2\left(\frac{x-s}{x\cancel{s}}\right)\cancel{s}\right)y = 2s$

i.e. $\left(1-\frac{2(x-s)}{x}\right)y = 2s$

$\therefore\ s = \frac{xy}{2(y-x)}$

So $t = \frac{x-s}{xs} = \dots = \frac{y-2x}{xy}$

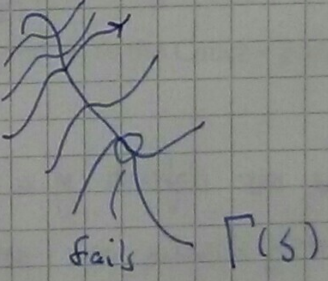
Hence $u = \frac{s^2}{1-ts^2} = \frac{x^2y^2}{y-x}\cdot\left(1-\frac{y-2x}{xy}\cdot\frac{x^2y^2}{4(y-x)^2}\right)^{-1}$

$\dots = \frac{x^2y^2}{4(y-x)^2-xy(y-2x)}$

Solution not defined on curve $4(x-y)^2 = xy(y-2x)$ in $\mathbb{R}^2$

(b) Transversality condition:

Need direction of characteristic curve at intersection with initial curve to be not parallel (transverse) when projected onto (x,y) plane.



Projection of initial curve $(s, 2s, s^2)$ onto (x,y) plane is in direction $(1,2)$ $\forall s$.

For characteristic curve, ~~direction~~ $(x(t,s), y(t,s), u(t,s))$

Direction is given by $(x_t(0,s), y_t(0,s))$

i.e. $(x(0,s)^2, y(0,s)^2) = s^2(1,4)$

transversality condition fails at $s=0$.

(4) $\Omega \subset \mathbb{R}^n$ open, bounded, $u \in C^1(\Omega)$, $\varphi \in C_c^\infty(\Omega)$.

<u>Show</u>: $\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, dx = -\int_\Omega \underbrace{\frac{\partial u}{\partial x_i}}_{\text{strong derivative}} \varphi \, dx$

Let B be a ball large enough so that $\Omega \subset B$. Then ∂B is C^1 (smooth!), $\varphi = 0$ on $\partial\Omega$. Hence $u\varphi = 0$ on $\partial\Omega$. Extend φ and φu continuously by zero to all of $\bar{B}$. Then $u\varphi = 0$ on ∂B, and $u\varphi \in C^1(\bar{B})$.

Hence, by Gauss-Green,

$$\int_\Omega \frac{\partial}{\partial x_i}(u\varphi) \, dx \underset{\substack{u\varphi=0 \\ \text{off } \Omega}}{=} \int_B \frac{\partial}{\partial x_i}(u\varphi) \, dx$$

$$= \int_{\partial B} \underbrace{(u\varphi)}_{=0 \text{ on } \partial B} \nu^{(i)} \, dS = 0$$

Hence, differentiating LHS (classically),

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, dx + \int_\Omega \frac{\partial u}{\partial x_i} \varphi \, dx = 0$$

(*) This Motivates our definition of a weak derivative (and shows strong derivatives are weak derivatives)